AUTHOR: Buongiorno Nardelli, Marco
TITLE: Compositional design and pitch class set operations: integrated tools for composition and analysis
KEYWORDS: compositional design, pitch class operations, harmonic content, analysis

Marco Buongiorno Nardelli
North Carolina State University
Department of Physics
Raleigh, NC 27695
mbnardelli@ncsu.edu

ABSTRACT. In this paper, I will introduce an approach to composition and analysis founded on the integration of the notions of compositional design and pitch class set operations. The approach is based on the following four elements: (i) classification of the pitch class set, (ii) analysis of the set's harmonic content, (iii) conception of the compositional design and (iv) definition of the rules for the manipulation and transformation of the pitch class set.
Although some of these ideas will be exemplified with an analysis of a fragment of the counterpoint 1 from *The Art of Fugue* by J.S. Bach, the approach outlined in this paper provides a comprehensive and very general framework that encompasses any music expression, irrespective of style, genre, musical culture, period and artistic perception.

Introduction

[1.1] The classification of pitches in pitch class sets[1] offers a comprehensive approach to composition that frees the composer beyond the boundaries of any style or esthetic conception and allows the analysis of any musical piece according to objective factors. However, the sheer number of choices that are left to the individual decision can become overwhelming and the need of an organizational superstructure (as a theory of grand-unification) becomes apparent in order to maintain a coherent artistic direction or produce a sensible analysis beyond randomness and chaos (which, in itself, can be considered as a perfectly viable artistic choice, nonetheless).

What follows is an account of my personal approach to gain artistic coherence either in analyzing or composing a piece of music. Although the different topics treated here have been individually the subject of extensive investigations, the present paper summarizes my effort to understand and synthesize the different angles and perspectives in a coherent and general approach to composition and analysis.

[1.2] This approach is based on the following basic concepts:

1. <u>classification of the pitch class sets</u>. Although Forte's original classification is at the basis of any modern theory of pitch space relations, I will be mainly concerned on the alternative classification based on the Linear Interval Succession Vector (LISV), corresponding to the cyclic interval succession vector of stride 1 as introduced by Morris;[3]
2. <u>analysis of the harmonic content of each set</u>. This is what I consider to be the basic palette associated with any particular set of pitch classes: in general terms it is the ensemble of all the subsets of a specific class set. In particular and in a post-pan-tonal perspective, it can be defined as the ensemble of all principal chords (triads, tetrachords, pentachords hexachord etc. in any flavor, Maj, min, dim, etc.) that can be constructed with that pitch class set;[2]
3. <u>conception of the compositional design</u>. I consider compositional design as the skeleton of any musical composition.[3] In the following I will be mainly concerned with harmonic design, but rhythmic design is as important and should always be part of the creative or analytic thinking;
4. <u>manipulation and transformation of the pitch class sets and classification of the resulting combinations in chord progressions</u>. These are fundamental steps in compositional design that in my opinion have been kept somewhat separated but that are just a different aspect of the same problem. On one hand, algebraic operations on sets and classification of their symmetry properties have been a constant of atonal theory for decades[1,3,4] while on the other hand, a classification of chords in terms of their geometric properties and their resulting progressions is only relatively recent.[5] However, the cognitive appreciation of a piece of music cannot come from either of the two, but only by way of an integration of the different aspects into a single creative, analytical or emotional process.

In what follows I will elaborate on the above points and provide an example of a compositional and analytical process based on these ideas.

Classification of the pitch class sets

[2.1] The classification of pitch class sets in ordered ensembles is common knowledge and I will only review the basic concepts and definitions without demonstrating any of the important properties of the sets (see the excellent notes from Paul Nelson, 2003[6] for a simple and short primer or refer to textbooks like Rahn[7] or Forte[1] for a deeper treatise).

[2.2] Pitch class. A pitch class (pc) is the equivalence class of all pitches that are exactly octaves apart: for example pc 1 is the ensemble of all C# (and D♭, B##, etc.) in any octave.
[2.3] Pitch class set. Any unordered collection of pc. In a pitch class set the disregard for all distinctions besides pc content makes it easier to see relations that have to do only with the latter. This becomes relevant when analyzing the harmonic content of a pc set.
[2.4] Classification. pc sets are classified easily into types according to the number of the elements N comprising the set (cardinality). Types are equivalence classes that partition the domain of all the possible pc sets. Each type is represented by one of its members in its most normal form. The members of a set of pc can be listed in any order without changing the identity of the set. However, in order to obtain a universal classification, the sets are usually listed in their "normal form". An ordering that increases from left to right within an octave and occupies the least possible interval (set most packed to the left).
[2.5] Sets can be classified according to their reduction to normal form, in particular, two pc sets are equivalent if their normal form is the same (modulo an arbitrary mod(12) transposition). Each normal form is characterized by a specific interval content (number of different intervals that can be formed by the elements of the pc set), which can be uniquely specified by determining the interval vector of a set (the count of all the unordered pc intervals between all pairs of notes of the set). The interval vector assigns to each un-equivalent pc set its own interval content but still does not permit a clear aural identification of the set (the "color" of the set, that is defined more specifically by the harmonic content rather than the interval content, or at least by the combination of the two). Un-equivalent sets have been classified and named by Forte in *The Structure of Atonal Music*.
[2.6] An alternative classification of pc sets, and one independent of individual pc choices, can be obtained using the definition of the *linear interval sequence vector* (LISV). The linear interval sequence vector (indicated here by numbers between {}), is the vector containing the linear sequence of intervals in an ordered (not necessarily normal ordered) pc set. In practice is the sequence of number of semitones that separate each element of the ordered set: as an example, the linear interval sequence vector of set 7-35 (tonal major scale) is the well-known series of "tones" and "semitones" {2,2,1,2,2,2,1}. The LISV is nothing but the cyclic interval succession vector of stride 1, $CINT_1$, as defined by Morris (p. 40*ff*).[3,8]
[2.7] The LISV of the inversion of a set is the retroversion of the elements of the original LISV (the array read backwards) modulo cyclic permutations (for some sets the two might coincide, as in the inverse of 7-35 (natural minor scale) that has the same LISV modulo a single step cyclic rotation {2,1,2,2,1,2,2}). This classification is independent of the choice of individual pitches and each LISV sequence is uniquely associated with a particular set type in the Forte's scheme.[9]
For the purpose of the following discussion, I am specifically concerned with the analysis of the harmonic content of a pc set (related to its interval vector and LISV), and subsequently, with the set of operators that manipulate and modify a pc set of a given type (associated with both interval vector and LISV).

Harmonic content analysis of a pc set

[3.1] The harmonic content of a pc set is the ensemble of all the subsets of the pc set of N elements that have cardinality < N. In particular, one can restrict the choice of the subset to represent specific artistic and musical directions (only minor and major trichords, only diminished tetrachords, etc.). The definition of the subset in the analysis of a musical piece is clearly dictated by the style and the period of the music, so, for instance, one might want to consider only chord combinations that obey the principle of the so-called “Common Practice” (CP). As an example, in Table I I display the harmonic content (in terms of “named” chords) of the set 7-35, the major scale with 0 as the lowest note (pc set=[0,2,4,5,7,9,11]).[10] See the Appendix for a comprehensive table of the properties of the 223 pc sets.

Table I. Harmonic content of set 7-35 [0,2,4,5,7,9,11]

	Trichords		Tetrachords		Pentachords		Hexachords
Maj	0 4 7	m(add2)	2 4 5 9	m6/9	2 5 9 11 4	m11	2 5 9 0 4 7
	5 9 0		9 11 0 4	m9	2 5 9 0 4		9 0 4 7 11 2
	7 11 2	m(add4)	2 5 7 9		9 0 4 7 11	Maj9#11	5 9 0 4 7 11
m	2 5 9		4 7 9 11	9	7 11 2 5 9		
	4 7 11		9 0 2 4	9♭5	2 5 9 0 4		
	9 0 4	add2	0 2 4 7		9 0 4 7 11		
dim	11 2 5		5 7 9 0	Maj9	0 4 7 11 2		
♭5	5 9 11		7 9 11 2		5 9 0 4 7		
add2	0 2 7	add4	0 4 5 7	6(add7)	7 11 2 4 5		
	2 4 9		7 11 0 2	6/9	0 4 7 9 2		
	5 7 0	m7♭5	11 2 5 9		5 9 0 2 7		
	7 9 2	m6	2 5 9 11		7 11 2 4 9		
	9 11 4	m7	2 5 9 0	9+	9 4 5 7 11		
add4	0 5 7		4 7 11 2	9sus4	2 7 9 0 4		
	2 7 9		9 0 4 7		7 0 2 5 9		
	4 9 11	6	0 4 7 9		9 2 4 7 11		
	7 0 2		5 9 0 2				
	9 2 4		7 11 2 4				
		7	7 11 2 5				
		Maj7	0 4 7 11				
			5 9 0 4				

Compositional design

[4.1] Following Morris,[3] the term compositional design denotes the abstract and un-interpreted composition of pc or pc sets that form the skeleton of a musical conception. From Morris: “Compositional designs are more akin to figured bass in Baroque continuo parts or the chord symbols used in lead sheets in jazz, in that such notations guide both composition and improvisation but, once mastered, do not, directly or indirectly, influence stylistic and personal choice”. Uncovering or creating the compositional design of a composition provides a general framework for interpretation of the pc material in a completely free of formal rules fashion.

[4.2] Compositional designs consists of an array of pc that can be ordered either horizontally (melody) or vertically (harmony) and provide the canvass for the construction and realization of the future musical ideas. The power of the approach lays in the generality of the tool and the versatility in providing the composer (or the music critic) with the basic “color palette” of the piece. Central to this conception is the fact that each compositional design can be realized in an infinite number of ways, in any style and can be constructed according to pre-imposed rules that are decided independently by the composer. In this respect, it is rather easy to encompass any musical structure from common practice to atonal and beyond and extend the approach to non-western and microtonal systems (provided that a proper classification is produced).

[4.3] As an example I would like to discuss the compositional design of the first counterpoint from *The Art of Fugue* by J.S. Bach.[11] For simplicity I will concentrate on 8 bars (9 through 16) as displayed in Fig. 1.

Figure 1. Original bars 9-16 of Counterpoint 1 from *The art of Fugue*.

Pruning the parts from the embellishments, retardations and anticipations and general melodic additions, one can derive the underlying compositional design (here I assume an harmonic rhythm of 2 harmonies per bar, following the development of the main theme, but other, more or less dense choices would also be perfectly appropriate). The results are summarized in Table II.

Table II. Compositional design of bars 9-16 of the Counterpoint 1 from *The art of Fugue*.

2 1	2 0	10 9	9 0	0 11	9 9	11 0	9 9
5 4	2 5	7 5	2 7	5 5	4 4	2 0	
				9 2	0 9	8 9	0 0
2 9	5 2	1 2	5 5	2 2	9 0	5 4	4 7

The mere enumeration of the compositional design still lacks an underlying general methodology that can produce a satisfying analysis or provide direction and focus to a new composition. In order to create such a coherent approach we need to introduce the notions of manipulation and transformation of the pitch class set and the classification of the resulting combinations in chord progressions.

Operators in the pc space.

[5.1] The diverse entities of the pc space (sets, segments, rows etc.) provide the raw material for the construction of any compositional design. However, the determination of the mutual relations between the different elements is at the core of any coherent design of a composition. These relations are obtained by the application of the ensemble of Twelve Tone Operators (TTOs), that is the set of algebraic operators that can be applied to any pc element. In what follows I will consider TTOs that operate on pc sets, since I consider the pc set as the fundamental element of compositional design.

[5.2] TTOs can be divided into different classes:

1. <u>non-singular close operators</u>, the ones that transform a pc set into a different one belonging to the same type (Forte number), i.e. they preserve cardinality and normal form. These are for

instance all the transpositions T_n ($T_n(x)=x+n \mod(12)$) and inversions T_nI ($T_nI(x)=-x+n \mod(12)$);

2. <u>non-singular open operators</u>, the ones that transform a pc set into a different one that might or might not be of the same type, i.e. they preserve cardinality but not necessarily the normal form. These are the multiplicative operators M5 (circle of fourths, M5(x)=5*x mod(12)) and M7 (circle of fifths, M7(x)=7*x mod(12)=M5I(x)), and the "relational" operators R_p^n, the operators that raise or lower the p-th pc of the (ordered) pc set by an integer (positive or negative) number n preserving the cardinality of the set (see Figure 2 for an illustration of the basic TTOs operationson a set);
3. <u>singular operators</u>, all the operators that do not preserve the cardinality of the set (reduction to sub-set or extrapolation to super-set).

T											
0	1	2	0	1	0	0	1	0	1	0	0
1	2	3	3	4	2	1	2	2	3	2	1
2	3	4	4	5	5	3	4	3	4	4	3
4	5	6	5	6	6	6	7	5	6	5	5
6	7	8	7	8	7	7	8	8	9	7	6
7	8	9	9	10	9	8	9	9	10	10	8
9	10	11	10	11	11	10	11	10	11	11	11
TI											
0	0	0	1	0	1	0	0	1	0	1	2
3	1	1	2	2	3	2	1	2	2	3	4
5	4	2	3	3	4	4	3	4	3	4	5
6	6	5	6	4	5	5	5	6	5	6	7
8	7	7	8	7	8	6	6	7	7	8	9
10	9	8	9	9	10	9	7	8	8	9	10
11	11	10	11	10	11	11	10	11	9	10	11
TM											
0	0	0	0	0	1	0	0	1	2	3	4
5	1	1	1	1	2	2	1	2	3	4	5
6	6	2	2	2	3	3	3	4	5	6	7
8	7	7	3	3	4	4	4	5	6	7	8
9	9	8	8	4	5	5	5	6	7	8	9
10	10	10	9	9	10	6	6	7	8	9	10
11	11	11	11	10	11	11	7	8	9	10	11
TMI											
0	1	2	3	4	0	0	1	0	0	0	0
1	2	3	4	5	5	1	2	2	1	1	1
2	3	4	5	6	6	6	7	3	3	2	2
3	4	5	6	7	7	7	8	8	4	4	3
4	5	6	7	8	8	8	9	9	9	5	5
6	7	8	9	10	9	9	10	10	10	10	6
7	8	9	10	11	11	10	11	11	11	11	11

Figure 2. Transformation of set 7-29 with transposition, inversion and multiplication TTOs all in ascending order.

[5.3] Of all the TTOs the non-singular close ones are most studied and for which important theorems and properties have been demonstrated. For the purpose of composition or analysis I think that the most fundamental aspects are the ones of symmetry and common-tone properties. The degree of symmetry of a pc set is the number of distinct operations that transform the set in itself (typical examples are set that are transpositionally or inversionally symmetrical, or that are symmetric for circle of fourth or fifths transformations) while common-tone properties address the degree of overlap between sets of same cardinality.[12] Multiplicative operations are also subject of specific properties that are interesting from the compositional point of view. The main aspect is that, contrary to close operators, multiplication operators act on the interval vector of the original set in well-defined ways[13] that can be exploited for the creation of nicely coherent compositional designs.

[5.4] For relational operators of the kind R_p^n the enumeration of the properties becomes a more complicated affair, since it involves finding relations between sets that might not be related by any simpler operations (transposition, inversion and multiplication). One is left with little guidance besides the classification of the subsets of a given set and their harmonic content properties. However, using the classification in terms of LISV, one can still enumerate a number of properties that can be used in the generation of related sets. In particular, a simple observation will suffice to demonstrate the following properties:

Property 1: *For any ordered pc set, any given subset (N→N-1) can be obtained by a singular relational operation that raises (or lowers) a single element of the set by n equal to the LIST component (interval) preceding (or following) the selected pc.*

Property 2: *For any ordered pc set, any given superset (N→N+1) can be obtained by adding pc within existing elements of the set when the LISV component is > 1.*

The full set of TTOs including the relational operators is what allows the realization of the extended concept of progression from one set to another and that gives direction and focus to the compositional design. The basic operations that can be combined to accomplish any compositional design are thus as follows:

1. Remain within a single pc set (in tonal language is the equivalent of remaining in the same key)
2. Modulation through transposition and/or inversion (this operation preserves the interval content (and LISV) and for the inversion produces a complementary harmonic content; in tonal language it creates the major/minor dichotomy)
3. Modulation through multiplications (M5 and or M7)
4. Modulation through relational operators
5. Modulation through reduction or extrapolation
6. Modulation through the introduction of (seemingly) unrelated sets (in the extended set space (any set of any cardinality) any two sets can be connected by a chain of single operations, that is, the cyclic group of all sets of any cardinality N that can be constructed with M elements (with M non necessarily=12 and N≤M) can be generated from a single element (0) by the recursive application of all the operations described above).[14]

[5.5] In order to demonstrate the generality and versatility of the approach, let's analyze the example of Fig. 1 and Table I with these concepts in mind. In this example Bach uses the set 7-32 (harmonic minor) in two transpositions related by a perfect fifth (7 semitones): in tonal language it uses Dm(harmonic) in the first four bars and Am(harmonic) in the second four. Throughout Counterpoint I, Bach cleverly uses a set of TTO operators (or at least, we can analyze his compositional process in this way) to move between two set centers: 7-32 as in the example above and 7-35 in the Dm(natural) and DMajor, thus using a combination of transposition, inversion and relational operators to span the whole space of his compositional design. Fig. 2 graphically summarizes the relations between the different pc sets.

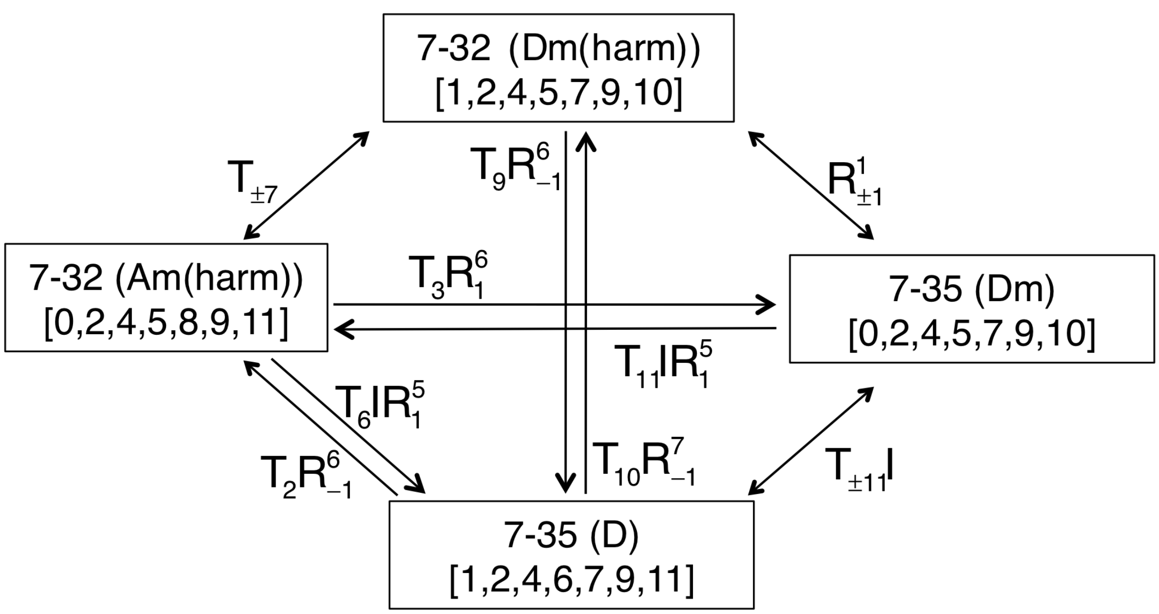

Figure 2. pc sets used in Counterpoint I and their mutual relations in terms of TTOs.

Note that 7-32 and 7-35 are related by the simplest relational operation, one pitch moved by one semitone and that the transformations involved in this graph are all reduced to a simple canonical form.[14]

Concluding remarks

[6.1] The approach outlined in this paper provides a comprehensive and very general framework that encompasses any music expression, irrespective of style, genre, musical culture, period and artistic perception. It provides the composer and the theorist with a multifaceted toolset that can be used at any degree of complexity. Moreover, it helps to elucidate the internal structure of the pc space and to open the road to explorations far beyond any specific set limitation. It suffices to observe that most of "classical" western music is founded on the exploitation of merely a handful of sets (mostly 7-35 and 7-32 corresponding to the major, natural minor and harmonic minor scales) and their mutual relations. Within any choice of set combination one can and should establish "compositional rules" to further clarify creative thinking. Such a set of rules can be, for instance, the laws of tonal harmony or the principles of 12-tone serialism, depending on the piece to analyze of the artistic directions that the composer wishes to follow. What this integrated approach does is to provide a framework where any compositional design can be accommodated and accounted for.

Acknowledgements

[7.1] The author wishes to thank Allen Anderson and Alan Shockley for enlightening discussions and inspiring guidance. This paper is dedicated to *Maestro* Pablo Colino for having disclosed to me the magic of numbers in music at a very early age.

Appendix.

Set table with Forte classification number, normal form, interval vector, LISV and the triadic harmonic content (number of major, minor, diminished and augmented triads). I use the convention that A=10, B=11, C=12.

Forte number	Normal form	interval vector	LISV	Maj	min	dim	aug
(1-1)	[0]	<1000000>	{0}	0	0	0	0
(2-1)	[01]	<2100000>	{1B}	0	0	0	0
(2-2)	[02]	<2010000>	{2A}	0	0	0	0
(2-3)	[03]	<2001000>	{39}	0	0	0	0
(2-4)	[04]	<2000100>	{48}	0	0	0	0
(2-5)	[05]	<2000010>	{57}	0	0	0	0
(2-6)	[06]	<2000001>	{66}	0	0	0	0
(3-1)	[012]	<3210000>	{11A}	0	0	0	0
(3-2)	[013]	<3111000>	{129}	0	0	0	0
(3-3)	[014]	<3101100>	{138}	0	0	0	0
(3-4)	[015]	<3100110>	{147}	0	0	0	0
(3-5)	[016]	<3100011>	{156}	0	0	0	0
(3-6)	[024]	<3020100>	{228}	0	0	0	0
(3-7)	[025]	<3011010>	{237}	0	0	0	0
(3-8)	[026]	<3010101>	{246}	0	0	0	0
(3-9)	[027]	<3010020>	{255}	0	0	0	0
(3-10)	[036]	<3002001>	{336}	0	0	1	0
(3-11)	[037]	<3001110>	{345}	0	1	0	0
(3-12)	[048]	<3000300>	{444}	0	0	0	3
(4-1)	[0123]	<4321000>	{1119}	0	0	0	0
(4-2)	[0124]	<4221100>	{1128}	0	0	0	0
(4-3)	[0134]	<4212100>	{1218}	0	0	0	0
(4-4)	[0125]	<4211110>	{1137}	0	0	0	0
(4-5)	[0126]	<4210111>	{1146}	0	0	0	0
(4-6)	[0127]	<4210021>	{1155}	0	0	0	0
(4-7)	[0145]	<4201210>	{1317}	0	0	0	0
(4-8)	[0156]	<4200121>	{1416}	0	0	0	0
(4-9)	[0167]	<4200022>	{1515}	0	0	0	0
(4-10)	[0235]	<4122010>	{2127}	0	0	0	0
(4-11)	[0135]	<4121110>	{1227}	0	0	0	0
(4-12)	[0236]	<4112101>	{2136}	0	0	1	0
(4-13)	[0136]	<4112011>	{1236}	0	0	1	0
(4-14)	[0237]	<4111120>	{2145}	0	1	0	0
(4-15)	[0146]	<4111111>	{1326}	0	0	0	0
(4-16)	[0157]	<4110121>	{1425}	0	0	0	0
(4-17)	[0347]	<4102210>	{3135}	1	1	0	0
(4-18)	[0147]	<4102111>	{1335}	1	0	1	0
(4-19)	[0148]	<4101310>	{1344}	0	1	0	3
(4-20)	[0158]	<4101220>	{1434}	1	1	0	0
(4-21)	[0246]	<4030201>	{2226}	0	0	0	0
(4-22)	[0247]	<4021120>	{2235}	1	0	0	0
(4-23)	[0257]	<4021030>	{2325}	0	0	0	0
(4-24)	[0248]	<4020301>	{2244}	0	0	0	3
(4-25)	[0268]	<4020202>	{2424}	0	0	0	0
(4-26)	[0358]	<4012120>	{3234}	1	1	0	0
(4-27)	[0258]	<4012111>	{2334}	0	1	1	0
(4-28)	[0369]	<4004002>	{3333}	0	0	4	0
(4-29)	[0137]	<4111111>	{1245}	0	1	0	0
(5-1)	[01234]	<5432100>	{11118}	0	0	0	0
(5-2)	[01235]	<5332110>	{11127}	0	0	0	0
(5-3)	[01245]	<5322210>	{11217}	0	0	0	0
(5-4)	[01236]	<5322111>	{11136}	0	0	1	0
(5-5)	[01237]	<5321121>	{11145}	0	1	0	0

(5-6)	[01256]	<5311221>	{11316}	0	0	0	0
(5-7)	[01267]	<5310132>	{11415}	0	0	0	0
(5-8)	[02346]	<5232201>	{21126}	0	0	1	0
(5-9)	[01246]	<5231211>	{11226}	0	0	0	0
(5-10)	[01346]	<5223111>	{12126}	0	0	1	0
(5-11)	[02347]	<5222220>	{21135}	1	1	0	0
(5-12)	[01356]	<5222121>	{12216}	0	0	1	0
(5-13)	[01248]	<5221311>	{11244}	0	1	0	3
(5-14)	[01257]	<5221131>	{11325}	0	0	0	0
(5-15)	[01268]	<5220222>	{11424}	0	0	0	0
(5-16)	[01347]	<5213211>	{12135}	1	1	1	0
(5-17)	[01348]	<52t2320>	{12144}	1	1	0	3
(5-18)	[01457]	<5212221>	{13125}	1	0	1	0
(5-19)	[01367]	<5212122>	{12315}	0	1	1	0
(5-20)	[01568]	<5211231>	{14124}	1	1	0	0
(5-21)	[01458]	<5202420>	{13134}	1	2	0	3
(5-22)	[01478]	<5202321>	{13314}	1	1	1	3
(5-23)	[02357]	<5132130>	{21225}	0	1	0	0
(5-24)	[01357]	<5131221>	{12225}	0	1	0	0
(5-25)	[02358]	<5123121>	{21234}	1	1	1	0
(5-26)	[02458]	<5122311>	{22134}	0	1	1	3
(5-27)	[01358]	<5122230>	{12234}	2	1	0	0
(5-28)	[02368]	<5122212>	{21324}	1	0	1	0
(5-29)	[01368]	<5122131>	{12324}	1	0	1	0
(5-30)	[01468]	<5121321>	{13224}	0	1	0	3
(5-31)	[01369]	<5114112>	{12333}	0	1	4	0
(5-32)	[01469]	<5113221>	{13233}	1	2	1	0
(5-33)	[02468]	<5040402>	{22224}	0	0	0	3
(5-34)	[02469]	<5032221>	{22233}	1	1	1	0
(5-35)	[02479]	<5032140>	{22323}	1	1	0	0
(5-36)	[01247]	<5222121>	{11235}	1	0	1	0
(5-37)	[03458]	<5212320>	{31134}	1	1	0	3
(5-38)	[01258]	<5212221>	{11334}	1	1	1	0
(6-1)	[012345]	<6543210>	{111117}	0	0	0	0
(6-2)	[012346]	<6443211>	{111126}	0	0	1	0
(6-3)	[012356]	<6433221>	{111216}	0	0	1	0
(6-4)	[012456]	<6432321>	{112116}	0	0	0	0
(6-5)	[012367]	<6422232>	{111315}	0	1	1	0
(6-6)	[012567]	<6421242>	{113115}	0	0	0	0
(6-7)	[012678]	<6420243>	{114114}	0	0	0	0
(6-8)	[023457]	<6343230>	{211125}	1	1	0	0
(6-9)	[012357]	<6342231>	{111225}	0	1	0	0
(6-10)	[013457]	<6333321>	{121125}	1	1	1	0
(6-11)	[012457]	<6333231>	{112125}	1	0	1	0
(6-12)	[012467]	<6332232>	{112215}	1	0	1	0
(6-13)	[013467]	<6324222>	{121215}	1	1	2	0
(6-14)	[013458]	<6323430>	{121134}	2	2	0	3
(6-15)	[012458]	<6323421>	{112134}	1	2	1	3
(6-16)	[014568]	<6322431>	{131124}	1	2	0	3
(6-17)	[012478]	<6322332>	{112314}	1	1	1	3
(6-18)	[012578]	<6322242>	{113214}	1	1	1	0
(6-19)	[013478]	<6313431>	{121314}	2	2	1	3
(6-20)	[014589]	<6303630>	{131313}	3	3	0	6
(6-21)	[023468]	<6242412>	{211224}	1	0	1	3
(6-22)	[012468]	<6241422>	{112224}	0	1	0	3
(6-23)	[023568]	<6234222>	{212124}	1	1	2	0
(6-24)	[013468]	<6233331>	{121224}	1	1	1	3
(6-25)	[013568]	<6233241>	{122124}	2	1	1	0
(6-26)	[013578]	<6232341>	{122214}	2	2	0	0
(6-27)	[013469]	<6225222>	{121233}	1	2	4	0
(6-28)	[013569]	<6224322>	{122133}	1	1	4	3
(6-29)	[023679]	<6224232>	{213123}	1	1	4	0

(6-30)	[013679]	<6224223>	{123123}	0	2	4	0
(6-31)	[014579]	<6223431>	{131223}	3	1	1	3
(6-32)	[024579]	<6143250>	{221223}	2	2	0	0
(6-33)	[023579]	<6143241>	{212223}	1	2	1	0
(6-34)	[013579]	<6142422>	{122223}	1	1	1	3
(6-35)	[02468A]	<6060603>	{222222}	0	0	0	6
(6-36)	[012347]	<6433221>	{111135}	1	1	1	0
(6-37)	[012348]	<6432321>	{111144}	1	1	0	3
(6-38)	[012378]	<6421242>	{111414}	1	1	0	0
(6-39)	[023458]	<6333321>	{211134}	1	1	1	3
(6-40)	[012358]	<6333231>	{111234}	2	1	1	0
(6-41)	[012368]	<6332232>	{111324}	1	0	1	0
(6-42)	[012369]	<6324222>	{111333}	1	1	4	0
(6-43)	[012568]	<6322332>	{113124}	1	1	1	0
(6-44)	[012569]	<6313431>	{113133}	2	2	1	3
(6-45)	[023469]	<6234222>	{211233}	1	1	4	0
(6-46)	[012469]	<6233331>	{112233}	2	2	1	0
(6-47)	[012479]	<6233241>	{112323}	2	1	1	0
(6-48)	[012579]	<6232341>	{113223}	1	1	0	3
(6-49)	[013479]	<6224322>	{121323}	2	2	2	0
(6-50)	[014679]	<6224232>	{132123}	2	2	2	0
(7-1)	[0123456]	<7654321>	{1111116}	0	0	1	0
(7-2)	[0123457]	<7554331>	{1111125}	1	1	1	0
(7-3)	[0123458]	<7544431>	{1111134}	2	2	1	3
(7-4)	[0123467]	<7544332>	{1111215}	1	1	2	0
(7-5)	[0123567]	<7543342>	{1112115}	0	1	1	0
(7-6)	[0123478]	<7533442>	{1111314}	2	2	1	3
(7-7)	[0123678]	<7532353>	{1113114}	1	1	1	0
(7-8)	[0234568]	<7454422>	{2111124}	1	1	2	3
(7-9)	[0123468]	<7453432>	{1111224}	1	1	1	3
(7-10)	[0123469]	<7445332>	{1111233}	2	2	4	0
(7-11)	[0134568]	<7444441>	{1211124}	2	2	1	3
(7-12)	[0123479]	<7444342>	{1111323}	2	2	2	0
(7-13)	[0124568]	<7443532>	{1121124}	1	2	1	3
(7-14)	[0123578]	<7443352>	{1112214}	2	2	1	0
(7-15)	[0124678]	<7442443>	{1122114}	1	1	1	3
(7-16)	[0123569]	<7435432>	{1112133}	2	2	4	3
(7-17)	[0124569]	<7434541>	{1121133}	3	3	1	3
(7-18)	[0123589]	<7434442>	{1311123}	3	2	2	3
(7-19)	[0123679]	<7434343>	{1113123}	1	2	4	0
(7-20)	[0125679]	<7433452>	{1131123}	2	2	1	3
(7-21)	[0124589]	<7424641>	{1121313}	3	4	1	6
(7-22)	[0125689]	<7424542>	{1131213}	3	3	2	3
(7-23)	[0234579]	<7354351>	{2111223}	2	3	1	0
(7-24)	[0123579]	<7353442>	{1112223}	1	2	1	3
(7-25)	[0234679]	<7345342>	{2112123}	2	2	4	0
(7-26)	[0134579]	<7344532>	{1211223}	3	2	2	3
(7-27)	[0124579]	<7344451>	{1121223}	3	2	1	3
(7-28)	[0135679]	<7344433>	{1221123}	1	2	4	3
(7-29)	[0124679]	<7344352>	{1122123}	3	2	2	0
(7-30)	[0124689]	<7343542>	{1122213}	2	3	1	3
(7-31)	[0134679]	<7336333>	{1212123}	2	3	5	0
(7-32)	[0134689]	<7335442>	{1212213}	2	3	4	3
(7-33)	[012468A]	<7262623>	{1122222}	1	1	1	6
(7-34)	[013468A]	<7254442>	{1212222}	2	2	2	3
(7-35)	[013568A]	<7254361>	{1221222}	3	3	1	0
(7-36)	[0123568]	<7444342>	{1112124}	2	1	2	0
(7-37)	[0134578]	<7434541>	{1211214}	3	3	1	3
(7-38)	[0124578]	<7434442>	{1121214}	2	2	2	3
(8-1)	[01234567]	<8765442>	{11111115}	1	1	2	0
(8-2)	[01234568]	<8665542>	{11111124}	2	2	2	3
(8-3)	[01234569]	<8656542>	{11111133}	3	3	4	3

(8-4)	[01234578]	<8655552>	{11111214}	3	3	2	3
(8-5)	[01234678]	<8654553>	{11112114}	2	2	2	3
(8-6)	[01235678]	<8654463>	{11121114}	2	2	2	0
(8-7)	[01234589]	<8645652>	{11111313}	4	4	2	6
(8-8)	[01234789]	<8644563>	{11113113}	3	3	2	3
(8-9)	[01236789]	<8644464>	{11131113}	2	2	4	0
(8-10)	[02345679]	<8566452>	{21111123}	3	3	4	0
(8-11)	[01234579]	<8565552>	{11111223}	3	3	2	3
(8-12)	[01345679]	<8556543>	{12111123}	3	3	5	3
(8-13)	[01234679]	<8556453>	{11112123}	3	3	5	0
(8-14)	[01245679]	<8555562>	{11211123}	4	3	2	3
(8-15)	[01234689]	<8555553>	{11112213}	3	3	4	3
(8-16)	[01235789]	<8554563>	{11122113}	3	3	2	3
(8-17)	[01345689]	<8546652>	{12111213}	4	4	4	6
(8-18)	[01235689]	<8546553>	{11121213}	4	3	5	3
(8-19)	[01245689]	<8545752>	{11211213}	4	5	2	6
(8-20)	[01245789]	<8545662>	{11212113}	4	4	2	6
(8-21)	[0123468A]	<8474643>	{11112222}	2	2	2	6
(8-22)	[0123568A]	<8465562>	{11121222}	4	3	2	3
(8-23)	[0123578A]	<8465472>	{11122122}	4	4	2	0
(8-24)	[0124568A]	<8464743>	{11211222}	3	3	2	6
(8-25)	[0124678A]	<8464644>	{11221122}	2	2	4	6
(8-26)	[0134578A]	<8456562>	{12112122}	4	4	4	3
(8-27)	[0124578A]	<8456553>	{11212122}	3	4	5	3
(8-28)	[0134679A]	<8448444>	{12121212}	4	4	8	0
(8-29)	[01235679]	<8555553>	{11121123}	2	3	4	3
(9-1)	[012345678]	<9876663>	{111111114}	3	3	3	3
(9-2)	[012345679]	<9777663>	{111111123}	4	4	5	3
(9-3)	[012345689]	<9767763>	{111111213}	5	5	5	6
(9-4)	[012345789]	<9766773>	{111112113}	5	5	3	6
(9-5)	[012346789]	<9766674>	{111121113}	4	4	5	3
(9-6)	[01234568A]	<9686763>	{111111222}	4	4	3	6
(9-7)	[01234578A]	<9677673>	{111112122}	5	5	5	3
(9-8)	[01234678A]	<9676764>	{111121122}	4	4	5	6
(9-9)	[01235678A]	<9676683>	{111211122}	5	5	3	3
(9-10)	[01234679A]	<9668664>	{111121212}	5	5	8	3
(9-11)	[01235679A]	<9667773>	{111211212}	5	6	5	6
(9-12)	[01245689A]	<9666963>	{112112112}	6	6	3	9
(10-1)	[0123456789]	<A988884>	{1111111113}	6	6	6	6
(10-2)	[012345678A]	<A898884>	{1111111122}	6	6	6	6
(10-3)	[012345679A]	<A889884>	{1111111212}	7	7	8	6
(10-4)	[012345689A]	<A888984>	{1111112112}	7	7	6	9
(10-5)	[012345789A]	<A888894>	{1111121112}	7	7	6	6
(10-6)	[012346789A]	<A888885>	{1111211112}	6	6	8	6
(11-1)	[0123456789A]	<BAAAAA5>	{11111111112}	9	9	9	9
(12-1)	[0123456789AB]	<CCCCCC6>	{111111111111}	12	12	12	12

[1] A. Forte, *The structure of atonal music*, Yale University Press, New Haven (1973)
[2] This definition is still perfectly valid in other pc spaces provided that a proper definition of chords is given in terms of relative intervals within the pc set (for instance, the complete microtonal space (that is the aggregate) of the Arabic tradition is the set of pc from 0 to 19, from which we can draw to define all possible chordal spaces (subsets)).
[3] R. Morris, *Composition with pitch-classes: a theory of compositional design*, Yale University Press, New Haven (1987)
[4] D. Lewin, *Generalized musical intervals and transformations*, Yale University Press, New Haven (1987)
[5] D. Tymoczko et al., The geometry of musical chords, Science **313,** 72 (2006)
[6] See ComposerTools.com or PNelsonComposer.com, an excellent resource on set theory in music and a principal inspiration and common tool for my work.
[7] J. Rahn, *Basic atonal theory*, Shirmer, New York (1980)
[8] R. Chrisman, Describing Structural Aspects of Pitch-Sets Using Successive-Interval Arrays, J. of Music Theory, **21**, 1 (1977)
[9] Interestingly enough, the classification of sets using the LISV lifts the degeneracy between Z-invariant sets, that is, it separates into independent categories sets that would have otherwise the same interval vector. For example, sets 4-Z15 [0,1,4,6] and 4-Z29 [0,1,3,7] have the same interval vector ⟨4,1,1,1,1,1,1⟩, but different LISV: {1,3,2,6} and {1,2,4,5} respectively.
[10] Note that here I extend the definition of "harmonies" to chords that would not have been considered as such in the CP period.
[11] J.S. Bach, *Die Kunst der Fugen* BWV 1080 , Bärenreiter Kassel, Basel (1956)
[12] The common-tone theorem for T_n states that the number of pc in common between a set and any of its non-tritone transposition equals the multiplicity of the interval of that transposition (tritone counts double). Similar concepts apply to the common-tone theorem for T_nI (See Rhan, pag. 107ff).
[13] Typically, M5 and M7 when operating on a pair of pc, they operate identically on their interval.
[14] The introduction of the relational operators extends the definition of canonical operator as in Morris, p. 43*ff*. A canonical operator is an operator where T is performed after M and/or R, and all composite operators can be reduced in canonical form.